

A SUFFICIENT CONDITION FOR TOTAL MONOTONICITY: CORRECTIONS

BY
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In the expression for μ_n in the proof of Theorem 5 [2, p. 315] the term $\Gamma(n+b)$ was incorrectly written as $\Gamma(n+\alpha)$. This paper corrects that mistake. I am indebted to Dr. S. K. Basu for bringing the error to my attention. The reader is referred to [2] for pertinent notation and definitions. Theorem 1 of this paper replaces Theorem 5 of [2].

THEOREM 1. (i) For $-1 < \alpha < 0$, $2b \geq 3 - \alpha$, C_b^α t.s. H^α .

(ii) For $-1 < \alpha < 0$, $1 < b \leq 1 - \alpha$, H^α t.s. C_b^α .

(iii) For $0 < \alpha < 1$, $2b \geq 3 - \alpha$, H^α t.s. C_b^α .

(iv) For $\alpha > 1$, $0 < a \leq (3 - \alpha)/2$, C_a^α t.s. H^α .

Let

$$\mu(t) = \frac{\Gamma(b+\alpha)\Gamma(t+b)(t+1)^\alpha}{\Gamma(b)\Gamma(t+b+\alpha)} = e^{\sigma(t)}.$$

Using the series expansion for the logarithmic derivative of the gamma function,

$$\begin{aligned} (-1)\sigma'(t) &= \frac{\Gamma'(t+b+\alpha)}{\Gamma(t+b+\alpha)} - \frac{\Gamma'(t+b)}{\Gamma(t+b)} - \frac{\alpha}{t+1} \\ &= \sum_{m=0}^{\infty} \left(\frac{1}{m+t+1} - \frac{1}{m+t+b+\alpha} \right) \\ (1) \quad &\quad - \sum_{m=0}^{\infty} \left(\frac{1}{m+t+1} - \frac{1}{m+t+b} \right) - \frac{\alpha}{t+1} \\ &= \sum_{m=0}^{\infty} \left(\frac{1}{m+t+b} - \frac{1}{m+t+b+\alpha} \right) - \alpha \sum_{m=0}^{\infty} \left(\frac{1}{m+t+1} - \frac{1}{m+t+2} \right) \\ &\quad (-1)\sigma'(t) = \alpha \sum_{m=0}^{\infty} f_m(t) g_m(t), \end{aligned}$$

where

$$(2) \quad 1/g_m(t) = (m+t+b)(m+t+b+\alpha)(m+t+1)(m+t+2),$$

and

$$\begin{aligned} (3) \quad f_m(t) &= (m+t+1)(m+t+2) - (m+t+b)(m+t+b+\alpha) \\ &= (3-2b-\alpha)(m+t) + 2 - b(b+\alpha). \end{aligned}$$

Replacing b in (3) by $(3-\alpha)/2$, then, from (1),

$$(4) \quad (-1)\sigma'(t) = \frac{\alpha(\alpha^2-1)}{4} \sum_{m=0}^{\infty} g_m(t).$$

From (2) it is clear that $\text{sgn } g_m^{(n)}(t) = (-1)^n$ for all $t \geq 0$. From (4) we have, for all $t \geq 0$,

$$(-1)^n \sigma^{(n)}(t) \geq 0 \quad \text{for } -1 < \alpha < 0 \text{ or } \alpha > 1,$$

$$(-1)^{n+1} \sigma^{(n+1)}(t) \geq 0 \quad \text{for } 0 < \alpha < 1,$$

and (i), (iii), and (iv) have been proved for equality.

To finish the proof of (i), for $-1 < \alpha < 0$, $c \geq b$, C_c^α t.s. C_b^α [2, p. 313, Theorem 2(ii)]. Since t.s. is transitive, (i) is now proved. The proofs of (iii) and (iv) are completed using [2, p. 313, Theorem 2(iii)].

To prove (ii), let $b = 1 - \alpha$ in (3). Then (4) becomes

$$(5) \quad (-1)\sigma'(t) = \alpha(1+\alpha) \sum_{m=0}^{\infty} h_m(t),$$

where $h_m(t) = (m+t+1)g_m(t)$. Clearly $\text{sgn } h_m^{(n)}(t) = (-1)^n$ for $t \geq 0$. Therefore, from (5),

$$(-1)^{n+1} \sigma^{(n)}(t) \geq 0 \quad \text{for all } t \geq 0, \quad -1 < \alpha < 0;$$

i.e., H^α t.s. C_b^α for $b = 1 - \alpha$. The proof is completed by again using [2, p. 313, Theorem 2(ii)].

Theorem 1 leaves unanswered the question of other comparisons for the remaining parameter values of a and b . This question is settled by the negative results of the following theorem.

THEOREM 2. (i) For $-1 < \alpha < 0$, $1 < b < (3-\alpha)/2$, C_b^α n.t.s. H^α .

(ii) For $-1 < \alpha \leq \gamma < 0$, $b > 1 - \alpha$, H^γ n.t.s. C_b^α .

(iii) For $0 < \alpha < 1$, $1 < b < (3-\alpha)/2$, H^α n.t.s. C_b^α .

(iv) For $0 < \alpha \leq \beta < 1$, $b > 1$, C_b^β n.t.s. H^α .

(v) For $\alpha > 1$, $(3-\alpha)/2 < a < 1$, C_a^α n.t.s. H^α .

(vi) For $1 < \alpha \leq \beta$, $0 < a < 1$, H^β n.t.s. C_a^α .

We shall first prove (i), (iii), and (v). From (2), $g_m(t) > 0$ for all $t \geq 0$, $m = 0, 1, 2, \dots$. If we let $c = 3 - 2b - \alpha$, then $c > 0$ for the values of b stated in (i) and (iii) and $c < 0$ for the values stated in (v), with b replaced by a . From (3), for all t sufficiently large, $f_n(t) > 0$ for $|\alpha| < 1$ and $f_n(t) < 0$ for $\alpha > 1$; $n = 0, 1, 2, \dots$. Therefore $(-1)\sigma'(t)$ in (1) is negative for $-1 < \alpha < 0$ or $\alpha > 1$ and positive for $0 < \alpha < 1$.

To prove the remaining parts we shall show that there exists a positive integer k for which $(-1)^k \mu^{(k)}(t) < 0$ for the moment function under consideration. The procedure will be to examine the coefficient of t^k in an infinite series expansion for $\mu(t)$ that is valid for $0 < t < 1$, and to use the fact that the sign of the k th derivative

of $\mu(t)$ is determined by the sign of the coefficient of t^k for values of t sufficiently small.

Proof of (ii). Let $\beta = -\alpha$, $\delta = -\gamma$, $0 < t < 1$, and define

$$\mu(t) = \Gamma(b)\mu_1(t)\mu_2(t)/\Gamma(\beta)\Gamma(b-\beta),$$

where $\mu_2(t) = (1+t)^\delta$, and

$$\begin{aligned}\mu_1(t) &= \frac{\Gamma(t+b-\beta)\Gamma(\beta)}{\Gamma(t+\beta)} = \int_0^1 u^{t+b-\beta-1}(1-u)^{\beta-1} du \\ &= \int_0^1 \left(u^{t+b-\beta-1} - (\beta-1)u^{t+b-\beta} + \dots \right. \\ &\quad \left. + \frac{(\beta-1)(\beta-2)\cdots(\beta-n)u^{t+b+n-\beta-1}}{n!} + \dots \right) du \\ &= \frac{1}{t+b-\beta} + \sum_{n=1}^{\infty} \frac{(1-\beta)(2-\beta)\cdots(n-\beta)}{n!(t+b-\beta+n)},\end{aligned}$$

the term by term integration being justified as in [1].

Expanding $(t+b-\beta+n)^{-1}$ in powers of t we have

$$\begin{aligned}\mu_1(t) &= \frac{1}{t+b-\beta+n} + \sum_{n=1}^{\infty} \frac{(1-\beta)(2-\beta)\cdots(n-\beta)}{n!} \sum_{k=0}^{\infty} \frac{(-1)^k t^k}{(b-\beta+n)^{k+1}} \\ &= \frac{1}{t+b-\beta+n} + \sum_{k=0}^{\infty} (-1)^k t^k \sum_{n=1}^{\infty} \frac{(1-\beta)(2-\beta)\cdots(n-\beta)}{n!(b-\beta+n)^{k+1}}.\end{aligned}$$

The inversion of the order of summation can be justified as in [1, p. 455].

Now let

$$C_{k+1}(\beta) = \sum_{n=1}^{\infty} \frac{(1-\beta)(2-\beta)\cdots(n-\beta)}{n!(b-\beta+n)^{k+1}} \quad (k = 0, 1, 2, \dots),$$

and expand $(t+b-\beta)^{-1}$ in powers of t to obtain

$$\mu_1(t) = \sum_{k=0}^{\infty} (-1)^k [(b-\beta)^{-k-1} + C_{k+1}(\beta)] t^k.$$

If we expand $\mu_2(t)$ in powers of t , then the coefficients of t^k in the power series expansion for the product is

$$\begin{aligned}&(-1)^k [(b-\beta)^{-k-1} + C_{k+1}(\beta)] + (-1)^{k-1} [(b-\beta)^{-k} + C_k(\beta)] \delta \\ &+ \sum_{r=2}^k (-1)^{k-r} [(b-\beta)^{-k+r-1} + C_{k-r+1}(\beta)] \frac{(-1)^{r-1} \delta(1-\delta)\cdots(r-1-\delta)}{r!} \\ &= (-1)^k (b-\beta)^{-k-1} \left[1 - (b-\beta) - \delta \sum_{r=2}^k \frac{(1-\delta)(2-\delta)\cdots(r-1-\delta)(b-\beta)^r}{r!} \right. \\ &\quad \left. + (b-\beta)^{k+1} \left\{ C_{k+1}(\beta) - \delta C_k(\beta) - \delta \sum_{r=2}^k \frac{(1-\delta)\cdots(r-1-\delta)C_{k-r+1}(\beta)}{r!} \right\} \right].\end{aligned}$$

Since $b - \beta = b + \alpha > 1$ by hypothesis, the first series diverges. Since $0 < \beta < 1$, $(b - \beta)^{k+1} C_{k+1}(\beta)$ is uniformly bounded in k . If the second series converges, then the quantity in brackets is negative for all k sufficiently large. If the second series diverges then, a fortiori, the quantity in brackets is negative for all k sufficiently large.

Proof of (iv). For $0 < t < 1$ now define

$$\mu(t) = \frac{\Gamma(b+\beta)\Gamma(t+b)(t+1)^\alpha}{\Gamma(b)\Gamma(t+b+\beta)} = \frac{\Gamma(b+\beta)}{\Gamma(b)} \mu_1(t)\mu_2(t),$$

where

$$\mu_2(t) = (1+t)^\alpha$$

and

$$\mu_1(t) = \frac{\Gamma(b)\Gamma(t+b)}{\Gamma(t+b+\beta)} = \int_0^1 u^{t+b-1}(1-u)^{\beta-1} du.$$

Using the same procedure as in the proof of (ii), we may write

$$\mu_1(t) = \sum_{k=0}^{\infty} (-1)^k [b^{-k-1} + d_{k+1}(\beta)] t^k,$$

where

$$d_{k+1}(\beta) = \sum_{n=1}^{\infty} \frac{(1-\beta)(2-\beta)\cdots(n-\beta)}{n!(b+n)^{k+1}}.$$

Expanding $\mu_2(t)$ in powers of t , the coefficient of t^k in the power series expansion for the product can be written in the form

$$\begin{aligned} & (-1)^k b^{-k-1} \left[1 - b\alpha - \alpha \sum_{r=2}^k \frac{(1-\alpha)(2-\alpha)\cdots(r-1-\alpha)b^r}{r!} \right. \\ & \left. + b^{k+1} \left\{ d_{k+1}(\beta) - \alpha d_k(\beta) - \alpha \sum_{r=2}^k \frac{(1-\alpha)(2-\alpha)\cdots(r-1-\alpha)}{r!} d_{k-r+1}(\beta) \right\} \right]. \end{aligned}$$

Since $b > 1$, the first series diverges. Since $1 - \beta > 0$, $b^{k+1} d_{k+1}(\beta)$ is uniformly bounded in k . Whether or not the second series converges or diverges the quantity in brackets will be negative for all values of k sufficiently large.

Proof of (vi). We first prove that H^β n.t.s. C_a^1 for $\beta > 1$, $0 < a < 1$. Note that $C_a^1 = \Gamma_a^1$. Let

$$\mu(t) = (t+a)/a(t+1)^\beta.$$

Then, for $0 < t < 1$,

$$\begin{aligned} \mu(t) &= (1+t/a)(1+t)^{-\beta} \\ &= (1+t/a) \left[1 + \sum_{k=1}^{\infty} \frac{(-1)^k \beta(\beta+1)\cdots(\beta+k-1)t^k}{k!} \right]. \end{aligned}$$

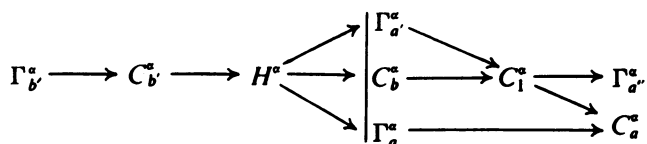
For $k > 1$, the coefficient of t^k is

$$\frac{(-1)^k \beta(\beta+1) \cdots (\beta+k-2)}{k!} [\beta+k-1-k/a].$$

Since $\beta > 1$, $a < 1$, the quantity in brackets will be negative for all k sufficiently large, and $\mu(t)$ is not totally monotone.

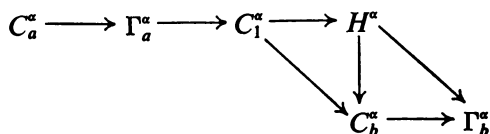
Now suppose H^β t.s. C_a^α for $1 < a \leq \beta$. From [2, p. 313, Theorem 2(i)], C_a^α t.s. C_a^1 for $\alpha > 1$. Since t.s. is transitive, H^β t.s. C_a^1 , a contradiction.

We conclude by listing a new total comparison table to replace the one on p. 316 of [2]. The arrow points toward the weaker method. Let $-1 < \alpha < 0$, $0 < a < 1$ and such that $a + \alpha > 0$, $0 < a'' \leq (\alpha + 1)/2 < 1 + \alpha \leq a' < 1$, $1 < b \leq 1 - \alpha < (3 - \alpha)/2 \leq b'$. Then

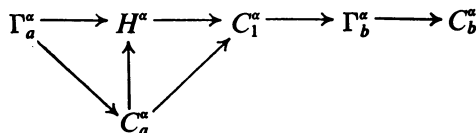


If $a'' \geq a$, then, of course $\Gamma_{a''}^\alpha$ t.s. C_a^α . If $a'' < a$, then $\Gamma_{a''}^\alpha$ and C_a^α are not totally comparable.

Let $0 < \alpha < 1$, $0 < a \leq (\alpha + 1)/2 < 1 < (3 - \alpha)/2 \leq b$. Then



Let $\alpha > 1$, $0 < a \leq (3 - \alpha)/2 < 1 < \alpha + 1 \leq 2b$. Then



REFERENCES

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